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Spacetime metric from linear electrodynamics II

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Abstract

Following Kottler, É.Cartan, and van Dantzig, we formulate the Maxwell equations in a metric independent form in terms of the field strength $F = (E, B)$ and the excitation $H = (\mathcal{D}, \mathcal{H})$. We assume a linear constitutive law between H and F. First we split off a pseudo-scalar (axion) field from the constitutive tensor; its remaining 20 components can be used to define a duality operator $^\#$ for 2-forms. If we enforce the constraint $^{\#\#} = -1$, then we can derive of that the conformally invariant part of the *metric* of spacetime. *file weimar16a.tex, 1999-11-24*

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1 Axiomatics of metric-free electrodynamics

We assume that spacetime is described by a smooth 4-dimensional manifold X and that it is possible to foliate X into 3-dimensional submanifolds which can be numbered by a monotonously increasing parameter σ. This could be called Axiom 0.

Our formulation of classical electrodynamics [24, 18, 9] is based on four axioms, three of which being of general validity and independent of the metric and/or affine structures of spacetime. Only in the context of the fourth axiom, the constitutive relation, the metric comes into play.

1.1 Axiom 1: Electric charge conservation

We assume the existence of a conserved electric current described by means of an odd 3-form J on X; for the exterior calculus involved, see [5]. Conservation of electric charge is a firmly established fact which basically can be verified by *counting* charged elementary particles inside a closed region. Mathematically, J is conserved when

$$\oint_{\Omega_3} J = 0\,, \qquad \partial\Omega_3 = 0\,, \tag{1}$$

where Ω_3 is an arbitrary closed 3-dimensional submanifold of the 4-manifold X. By de Rham's theorem, the current J is not only closed $dJ = 0$, but also exact, see [27, 23]. Thus the inhomogeneous Maxwell equation is a consequence of (1),

$$J = dH\,, \tag{2}$$

with H as the odd electromagnetic *excitation* 2-form. Note that H has an independent *operational* interpretation, see [9].

1.2 Axiom 2: Existence of the Lorentz force density

We introduce a field of frames e_α as reference system in X; by Greek letters $\alpha, \beta, \ldots = 0, 1, 2, 3$, we denote anholonomic or frame indices. The odd current 3-form J, together with the force density f_α (odd covector-valued 4-form), the notion of which is assumed to be known from mechanics, allows us to formulate the Lorentz force density as

$$f_\alpha = (e_\alpha \rfloor F) \wedge J\,. \tag{3}$$

Thereby the electromagnetic field strength F is defined as an even 2-form.

1.3 Axiom 3: Magnetic flux conservation

It is possible to count single quantized magnetic flux lines inside superconductors of type II. This suggests to take the conservation of magnetic flux as axiom 3,

$$\oint_{\Omega_2} F = 0\,, \quad \partial\Omega_2 = 0\,, \tag{4}$$

for an arbitrary closed submanifold Ω_2. As a consequence, we find the homogeneous Maxwell equation

$$dF = 0\,, \tag{5}$$

and the exactness of F, i.e., $F = dA$.

The Maxwell equations (2) and (5) are automatically diffeomorphism invariant, are independent of metric and connection (like the exterior *and* interior products), and are valid in this form in special and general relativity in arbitrary frames and in the post-Riemannian spacetimes of gauge theories of gravity, see [24].

1.4 Axiom 4: Constitutive law

To complete the formulation, a relation between H and F (and possibly J) is required, namely the constitutive law. We confine ourselves to a *linear* constitutive law ('linear electrodynamics'). In terms of the components of H and F in an arbitrary coordinate system x^i, it reads $(i, j \ldots = 0, 1, 2, 3)$,

$$H_{ij} = \frac{1}{4}\,\epsilon_{ijkl}\,\chi^{klmn}\,F_{mn}, \qquad \text{with} \qquad \chi^{ijkl} = -\chi^{jikl} = -\chi^{ijlk}\,, \tag{6}$$

where ϵ_{ijkl} is the Levi-Civita symbol and χ^{ijkl} an even tensor density of weight $+1$. Taking into account that the free field Lagrangian is of the form $L \sim H \wedge F$, we find $\chi^{ijkl} = \chi^{klij}$, leaving only 21 independent functions for χ^{ijkl}.

Moreover, one can decompose χ^{ijkl} according to

$$\chi^{ijkl} = f(x)\ \overset{\circ}{\chi}{}^{ijkl} + \alpha(x)\,\epsilon^{ijkl}\,, \qquad \text{with} \qquad \overset{\circ}{\chi}{}^{[ijkl]} \equiv 0\,. \tag{7}$$

Here $f(x)$ is a dimensionfull scalar function such that $\overset{\circ}{\chi}{}^{ijkl}$ is dimensionless. The pseudo-scalar constitutive function $\alpha(x)$ can be identified as an Abelian axion field. It was first discussed by Ni [15, 16, 17]. Its experimental bounds have been discussed by Carroll et al.[4]. Usually, however, an hypothetical axion is considered in the context of *non*-Abelian gauge theories, see [31, 32, 14]. Note that $\overset{\circ}{\chi}{}^{ijkl}$ has analogous algebraic symmetries and the same number of 20 independent components as a Riemannian curvature tensor:

$$\overset{\circ}{\chi}{}^{ijkl} = -\overset{\circ}{\chi}{}^{jikl} = -\overset{\circ}{\chi}{}^{ijlk} = \overset{\circ}{\chi}{}^{klij}\,, \qquad \overset{\circ}{\chi}{}^{[ijkl]} = 0\,. \tag{8}$$

2 Duality operator # and its closure

By means of axiom 4, a new duality operator can be defined acting on 2-forms on X. In components, an arbitrary 2-form $\Theta = \frac{1}{2}\Theta_{ij}\, dx^i \wedge dx^j$ is mapped into the 2-form ${}^{\#}\Theta$ by

$${}^{\#}\Theta_{ij} := \frac{1}{4}\epsilon_{ijkl}\,\overset{\circ}{\chi}{}^{klmn}\,\Theta_{mn}\,. \tag{9}$$

No metric is involved in this process. Now the linear law (6) can be written as

$$H = \left(f\,{}^{\#} + \alpha\right) F. \tag{10}$$

We postulate that the duality operator, applied twice, should, up to a sign, lead back to the identity. By this closure relation or the 'electric and magnetic reciprocity' [27], we can additionally constrain the 20 independent components of $\overset{\circ}{\chi}$ without using a metric. This appears to be a sufficient condition for the *non*existence of *birefringence* in vacuum, see [15, 16, 13, 17, 8, 12, 6]. Therefore, we impose

$${}^{\#\#} = -1\,. \tag{11}$$

The minus sign yields Minkowskian signature[4], whereas the condition ${}^{\#\#} = +1$ would lead to Euclidean or to the mixed signature $(+,+,-,-)$.

Seemingly Toupin [27] and Schönberg [25] were the first to deduce the conformally invariant part of a metric from relations like (9) and (11). This

[4] A different duality operator could be defined by ${}^{\widehat{\#}}\Theta_{ij} = \frac{f}{4}\epsilon_{ijkl}\,\overset{\circ}{\chi}{}^{klmn}\,\Theta_{mn}$. It would satisfy ${}^{\widehat{\#}\widehat{\#}} = -f^2$.

was rediscovered by Jadczyk [11], whereas Wang [30] gave a revised presentation of Toupin's results. A forerunner was Peres [20], see in this context also the more recent papers by Piron and Moore [21, 22]. It was recognized by Brans [1, 2] that, within general relativity, it is possible to define a duality operator in much the same way as presented above, see (9) and (11), and that from this duality operator the metric can be recovered. Such structures were subsequently discussed by numerous people, by Capovilla et al. [3], 't Hooft [10], Harnett [7], and Obukhov & Tertychniy [19], amongst others, see also the references given there.

It is convenient to adopt a more compact *bivector* notation by defining the indices $I, J, \ldots = (01, 02, 03, 23, 31, 12)$. Then $\overset{\circ}{\chi}{}^{ijkl}$ becomes the symmetric 6×6 matrix $\overset{\circ}{\chi}{}^{IK}$ and (11) reads

$$\overset{\circ}{\chi}{}^{IJ}\,\epsilon_{JK}\,\overset{\circ}{\chi}{}^{KL}\,\epsilon_{LM} = -\delta^I_M\,, \qquad \text{with} \qquad \epsilon_{IJ}\,\overset{\circ}{\chi}{}^{IJ} \equiv 0\,. \tag{12}$$

In terms of 3×3-matrices

$$\overset{\circ}{\chi}{}^{IJ} = \overset{\circ}{\chi}{}^{JI} = \begin{pmatrix} A & C \\ C^{\mathrm{T}} & B \end{pmatrix}, \quad \epsilon^{IJ} = \epsilon^{JI} = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \tag{13}$$

where $A = A^{\mathrm{T}}\,, B = B^{\mathrm{T}}$, and the superscript $^{\mathrm{T}}$ denotes transposition. The general non-trivial solution of the closure relation (12) is

$$\overset{\circ}{\chi}{}^{IJ} = \begin{pmatrix} pB^{-1} + qN & B^{-1}K \\ -KB^{-1} & B \end{pmatrix}. \tag{14}$$

Here B is a nondegenerate arbitrary *symmetric* 3×3 matrix (6 independent components B_{ab}), K an arbitrary *antisymmetric* matrix (3 independent components $K_{ab} =: \epsilon_{abc}\,k^c$), N the symmetric matrix with components $N^{ab} := k^a k^b$, and $q := -1/\det B$, $p := [\mathrm{tr}(NB)/\det B] - 1$. Thus, Eq.(14) subsumes 9 independent components.

3 Selfduality and a triplet of 2-forms

The duality operator $^{\#}$ induces a decomposition of the 6-dimensional space of 2-forms into two 3-dimensional invariant subspaces corresponding to the eigenvalues $\pm i$. The 2-form basis $dx^i \wedge dx^j$ or Θ^I decomposes into two 3-dimensional column vectors

$$\Theta^I = \begin{pmatrix} \beta^a \\ \gamma_b \end{pmatrix}, \quad a, b, \ldots = 1, 2, 3. \tag{15}$$

The self-dual basis $\overset{\text{(s)}}{\Theta}{}^I = \frac{1}{2}(\Theta^I - i\,{}^{\#}\Theta^I)$ decomposes similarly into

$$\overset{\text{(s)}}{\Theta}{}^I = \begin{pmatrix} \overset{\text{(s)}}{\beta}{}^a \\ \overset{\text{(s)}}{\gamma}{}_b \end{pmatrix} . \tag{16}$$

One of the 3-dimensional invariant subspaces can be spanned by, say, $\overset{\text{(s)}}{\gamma}$. Then $\overset{\text{(s)}}{\beta}$ can be expressed in terms of $\overset{\text{(s)}}{\gamma}$ according to $\overset{\text{(s)}}{\beta} = (i + B^{-1}K)B^{-1}\overset{\text{(s)}}{\gamma}$. Therefore $\overset{\text{(s)}}{\gamma}$ or, equivalently, the *triplet of 2-forms*

$$S^{(a)} := -(B^{-1})^{ab}\ \overset{\text{(s)}}{\gamma}{}_b\,, \tag{17}$$

subsume the properties of this invariant subspace. Each of the 2-forms carry 3 independent components, i.e., they add up to 9 components.

The information of the constitutive matrix $\overset{\text{o}}{\chi}{}^{IJ}$ is now encoded into the triplet of 2-forms $S^{(a)}$. The latter satisfy the completeness relation

$$S^{(a)} \wedge S^{(b)} = \frac{1}{3}\,(B^{-1})^{ab}\,(B)_{cd}\,S^{(c)} \wedge S^{(d)}\,. \tag{18}$$

4 Extracting the metric

Within the context of $SU(2)$ Yang-Mills theory, Urbantke [28, 29] was able to derive a 4-dimensional metric g_{ij} from a triplet of 2-forms satisfying a completeness condition of the type (18). This procedure also applies in our $U(1)$-case. Explicitly, the Urbantke formulas read

$$\sqrt{\det g}\; g_{ij} \;=\; -\,\frac{2}{3}\,\sqrt{\det B}\,\epsilon_{abc}\,\epsilon^{klmn}\,S^{(a)}_{ik}\,S^{(b)}_{lm}S^{(c)}_{nj}\,, \tag{19}$$

$$\sqrt{\det g} \;=\; -\,\frac{1}{6}\,\epsilon^{klmn}\,B_{cd}\,S^{(c)}_{kl}\,S^{(d)}_{mn}\,. \tag{20}$$

The $S^{(a)}_{ij}$ are the components of the 2-form triplet $S^{(a)} = S^{(a)}_{ij}\,dx^i \wedge dx^i/2$. If we substitute the $S^{(a)}$ into (19) and (20), we can display the metric explicitly in terms of the constitutive coefficients:

$$g_{ij} = \frac{1}{\sqrt{\det B}}\left(\begin{array}{c|c} \det B & -\,k_a \\ \hline -\,k_b & -\,B_{ab} + (\det B)^{-1}\,k_a\,k_b \end{array}\right). \tag{21}$$

Here $k_a := B_{ab}k^b = B_{ab}\,\epsilon^{bcd}K_{cd}/2$. One can verify that the metric in (21) has Minkowskian signature. Since the triplet $S^{(a)}$ is defined up to an arbitrary scalar factor, we obtain a *conformal* class of metrics.

5 Properties of the metric

5.1 Hodge duality operator

The inverse of (21) is given by

$$g^{ij} = \frac{1}{\sqrt{\det B}} \left(\begin{array}{c|c} 1-(\det B)^{-1} k_c k^c & -k^b \\ \hline -k^a & -(\det B)(B^{-1})^{ab} \end{array} \right) . \tag{22}$$

With the help of (21) and (22), we can define the Hodge duality operator * attached to this metric. In terms of the components of the 2-form F, we have

$$^\star F_{ij} := \frac{\sqrt{-g}}{2}\, \epsilon_{ijkl}\, g^{km} g^{ln} F_{mn}. \tag{23}$$

This equation can be rewritten, in analogy to (9), by defining the constitutive tensor $\overset{\mathrm{g}}{\chi}{}^{ijkl}$:

$$^\star F_{ij} = \frac{1}{4}\epsilon_{ijkl}\ \overset{\mathrm{g}}{\chi}{}^{klmn} F_{mn}\,, \quad \text{with} \quad \overset{\mathrm{g}}{\chi}{}^{ijkl} := \sqrt{-g}\left(g^{ik}g^{jl} - g^{jk}g^{il}\right) . \tag{24}$$

Note that $\overset{\mathrm{g}}{\chi}{}^{ijkl}$ is invariant under conformal transformations $g_{ij} \to e^{\lambda(x)} g_{ij}$; this takes care that only 9 of the possible 10 components of the metric can ever enter $\overset{\mathrm{g}}{\chi}{}^{ijkl}$.

In order to compare $\overset{\mathrm{g}}{\chi}{}^{ijkl}$ with the constitutive matrix χ^{IJ}, we put it in the form

$$\overset{\mathrm{g}}{\chi}{}^{IJ} = \left(\begin{array}{c|c} \overset{\mathrm{g}}{A} & \overset{\mathrm{g}}{C} \\ \hline \overset{\mathrm{g}}{C}{}^{\mathrm{T}} & \overset{\mathrm{g}}{B} \end{array} \right) . \tag{25}$$

Then straightforward calculations yield

$$\overset{\mathrm{g}}{A}{}^{ab} = g^{00}g^{ab} - g^{0a}g^{0b} = p\,(B^{-1})^{ab} + q\,N^{ab} = A^{ab}\,, \tag{26}$$

$$\overset{\mathrm{g}}{B}_{ab} = \frac{1}{4}\left(g^{ce}g^{df} - g^{de}g^{ef}\right)\epsilon_{acd}\,\epsilon_{efb} = B_{ab}\,, \tag{27}$$

$$\overset{\mathrm{g}}{C}{}^{a}{}_{b} = \frac{1}{2}\left(g^{0c}g^{ad} - g^{ac}g^{0d}\right)\epsilon_{bcd} = (B^{-1})^{ad}K_{db} = C^{a}{}_{b}. \tag{28}$$

Thus, $\overset{\mathrm{g}}{\chi}{}^{IJ} = \overset{\mathrm{o}}{\chi}{}^{IJ}$, i.e., the metric extracted allows us to write the original duality operator $^\#$ as Hodge duality operator, $^\# = {}^\star$. Therefore, the original constitutive tensor (7) can then be written as

$$\chi^{ijkl} = f(x)\sqrt{-g}\left(g^{ik}g^{jl} - g^{jk}g^{il}\right) + \alpha(x)\,\epsilon^{ijkl}\,. \tag{29}$$

This representation naturally suggests an interpretation of $f(x)$ as a scalar *dilaton* type field.[5]

5.2 Isotropy

Given a metric, we can define the notion of local isotropy. Let $T^{i_1\cdots i_p}$ be the contravariant coordinate components of a tensor field and $T^{\alpha_1\ldots\alpha_p} := e_{i_1}{}^{\alpha_1}\cdots e_{i_p}{}^{\alpha_p}\,T^{i_1\ldots i_p}$ its frame components with respect to an orthonormal frame $e_\alpha = e^i{}_\alpha\,\partial_i$. A tensor is said to be locally isotropic at a given point, if its frame components are invariant under a Lorentz rotation of the orthonormal frame. Similar considerations extend to tensor densities.

There are only two geometrical objects which are numerically invariant under (local) Lorentz transformations: the Minkowski metric $o_{\alpha\beta} = \mathrm{diag}(+1,-1,-1,-1)$ and the Levi-Civita tensor density $\epsilon_{\alpha\beta\gamma\delta}$. Thus

$$\mathcal{T}^{ijkl} = \phi(x)\,\sqrt{-g}\left(g^{ik}g^{jl} - g^{jk}g^{il}\right) + \varphi(x)\,\epsilon^{ijkl} \tag{30}$$

is the most general locally isotropic contravariant fourth rank tensor density of weight $+1$ with the symmetries $\mathcal{T}^{ijkl} = -\mathcal{T}^{jikl} = -\mathcal{T}^{ijlk} = \mathcal{T}^{klij}$. Here ϕ and φ are scalar and pseudo-scalar fields, respectively.

Accordingly, in view of (29), we have proved that the constitutive tensor (7) with the closure property (11) is *locally isotropic with respect to the metric* (21), see also [16] and [26].

6 Outlook

Developing the ideas of Kottler-Cartan-van Dantzig and following our previous work [18], we demonstrated that the general structure of classical electrodynamics is fundamentally independent of metric and connection.

The pseudo-Riemannian metric arises naturally in the context of *linear* electrodynamics from the duality operator $^\#$ constructed in terms of the constitutive coefficients and from its closure relation $^{\#\#} = -1$. At the same time, two *pre*metric objects emerge from the constitutive law: the pseudoscalar *axion* field $\alpha(x)$ and the scalar *dilaton* type field $f(x)$. They respect

[5] In the low-energy string models this factor is usually written as $f(x) = e^{-b\phi(x)}$ with a constant b and the dilation field $\phi(x)$.

all the axioms of electrodynamics (charge conservation, flux conservation, etc.).

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